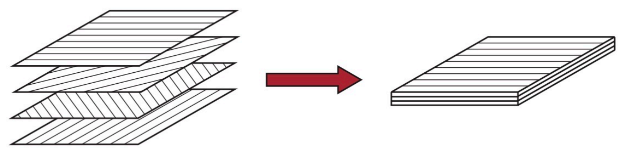
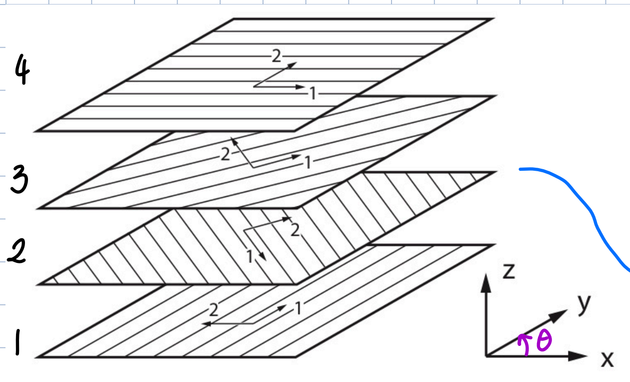
Laminate : combo of multiple laminae

↳ tailored structural properties

→ ply material properties and thickness
→ ply orientation wrt structural axes
→ ply stacking sequence



Layup Notation :



- Ply numbering bottom-up in +ve z
- angle θ defines orientation of ply material axes

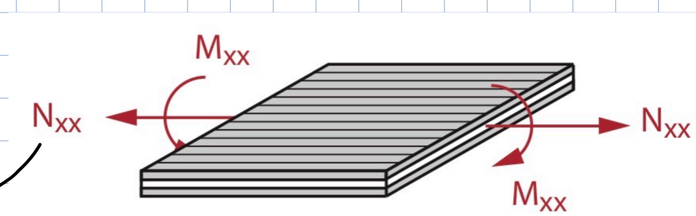
$[90/-45/45/0]$

- Repeating angles : θ_n
- Mirror-symmetry around x-axis : $\pm$ or $\mp$
- Mirror-symmetry around mid-plane : $[0/90/90/0] = [0/90]_s$

$[0/90/0] = [0/\overline{90}]_s$

Composite Plate Model :

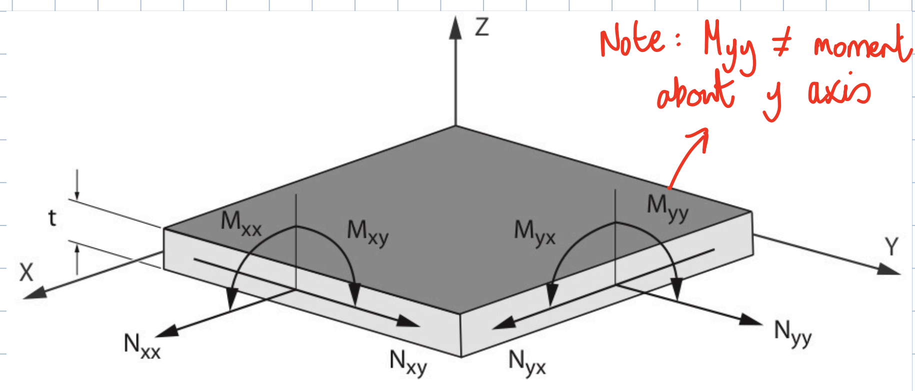
In-plane loads



Out-of-plane loads

Assumptions :

- Plys macro. homogenous & linear-elastic
- Perfectly bonded ∴ strain continuity
- Each ply in plane stress



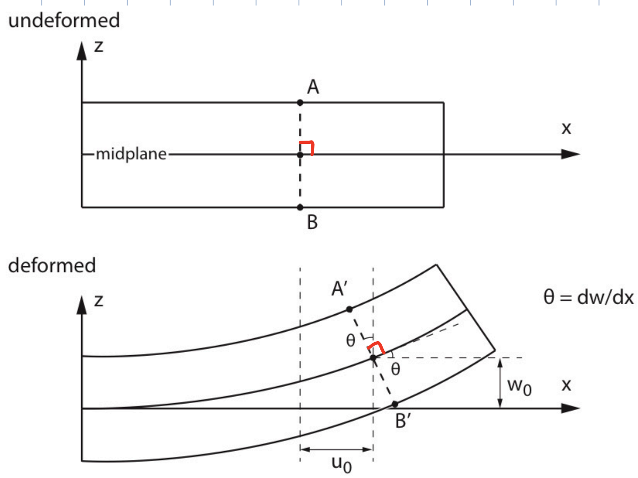
Note: $M_{yy} \neq$ moment about y axis

M & N defined per unit length:

$N = [N/m]$

$M = [N]$

Kirchoff - Love Plate Model : cross - sections remain straight



∴ x displacement anywhere along line $A'B'$:

$$U = U_0 - z\frac{\partial w_0}{\partial x}$$

and in y: $$V = V_0 - z\frac{\partial w_0}{\partial y}$$

→ Strains (from STM2 expressions):

strains at any point on the cross-section:

0 : midplane value

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{bmatrix} + z \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

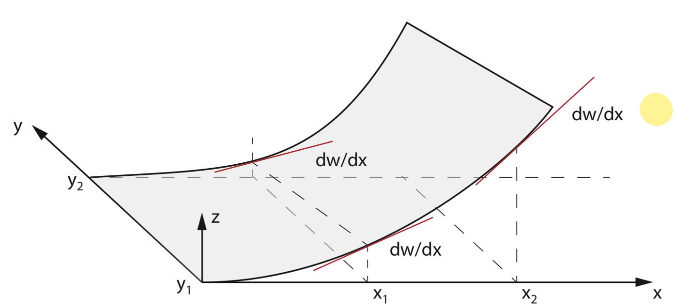
function of midplane strains and curvatures, and distance z

curvatures $(\kappa_{xx}, \kappa_{yy})$ and twist (κ_{xy}):

$$\boldsymbol{\kappa} = \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix} = - \begin{bmatrix} \partial^2 w_0/\partial x^2 \\ \partial^2 w_0/\partial y^2 \\ 2\partial^2 w_0/\partial x \partial y \end{bmatrix}$$

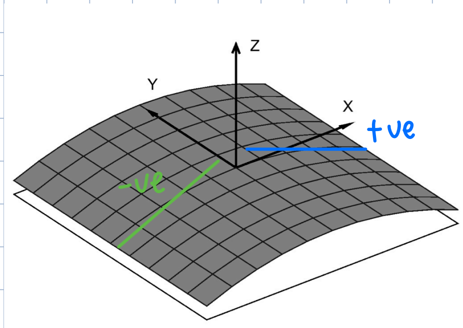
κ_{xx} rate of change in slope $\partial w/\partial x$ with respect to x

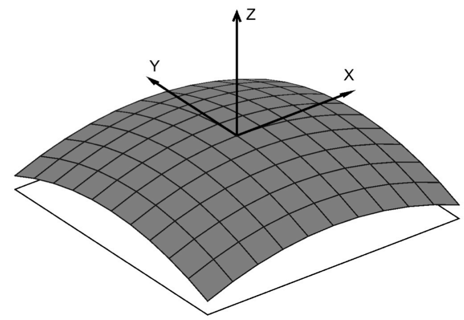
κ_{xy} rate of change of slope $\partial w/\partial x$ with respect to y



sign convention means this is -ve curvature

For κ_{xx} sign, imagine walking along x axis and note change in gradient etc.

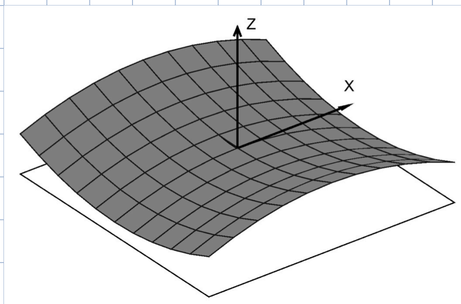


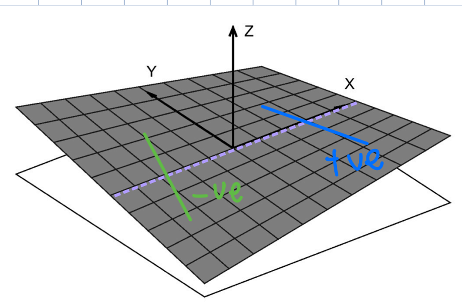


-ve to +ve $\therefore$ $\kappa_{xx} > 0$, $\kappa_{yy} = 0$, $\kappa_{xy} = 0$

$\kappa_{xx} > 0$
$\kappa_{yy} > 0$
$\kappa_{xy} = 0$

κ_{xy} ↳ walk along x and note change in y gradient (lean)



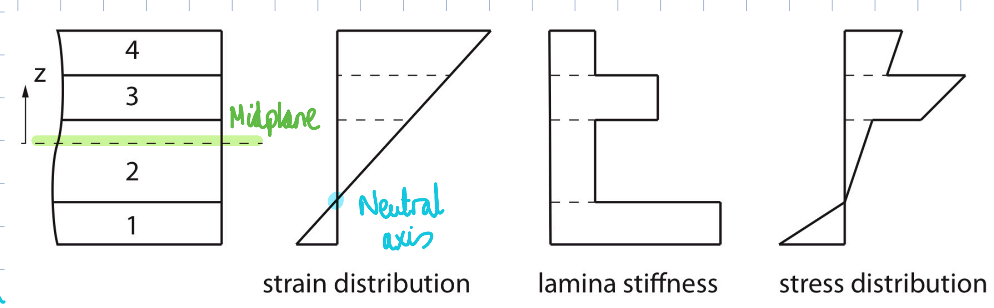


$\kappa_{xx} > 0$
$\kappa_{yy} < 0$
$\kappa_{xy} = 0$

$\kappa_{xx} = 0$
$\kappa_{yy} = 0$
$\kappa_{xy} > 0$

Midplane is NOT necessarily neutral axis

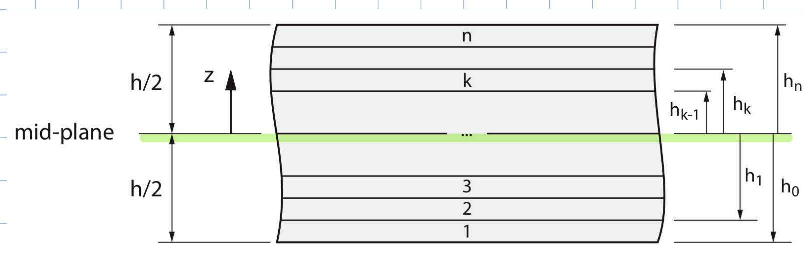
↳ point of 0 strain



Strain through cross-section continuous

Stiffness varies per laminae

$\therefore$ Stress is discontinuous across cross-section



Laminate labelled bottom-up.

Location h_k of each ply measured from midplane

$\boxed{\text{Ply thickness } t_k = h_k - h_{k-1}}$

→ each ply assummed to be in plane stress.

	ε^0_{xx}	ε^0_{yy}	γ^0_{xy}	κ_{xx}	κ_{yy}	κ_{xy}
N_{xx}	A_{11}	A_{12}	A_{16}	B_{11}	B_{12}	B_{16}
N_{yy}	A_{12}	A_{22}	A_{26}	B_{12}	B_{22}	B_{26}
N_{xy}	A_{16}	A_{26}	A_{66}	B_{16}	B_{26}	B_{66}
M_{xx}	B_{11}	B_{12}	B_{16}	D_{11}	D_{12}	D_{16}
M_{yy}	B_{12}	B_{22}	B_{26}	D_{12}	D_{22}	D_{26}
M_{xy}	B_{16}	B_{26}	B_{66}	D_{16}	D_{26}	D_{66}

- Each element of matrix symmetrical around diagonal as formed by $\bar{Q}$

Bending moment causes twisting curvature etc.

Twist is when xy rotation

- Direct stiffness terms
- Coupling - Poisson effects
- Extension - shear coupling terms

extensional stiffness matrix

A

Stack sequence irrelevant

$$A_{ij} = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k - h_{k-1}\right)$$

$$= \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k t_k$$

where t_k is thickness of k-th ply

coupling stiffness matrix

B

Stack sequence relevant

- In general describes in-plane out-of-plane coupling

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^2 - h_{k-1}^2\right)$$

$$= \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k - h_{k-1}\right) \frac{\left(h_k + h_{k-1}\right)}{2}$$

$$= \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k t_k \bar{z}_k$$

where $\bar{z}_k$ is distance to middle of k-th ply

bending stiffness matrix

D

Similar concept to A but for bending

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^3 - h_{k-1}^3\right)$$

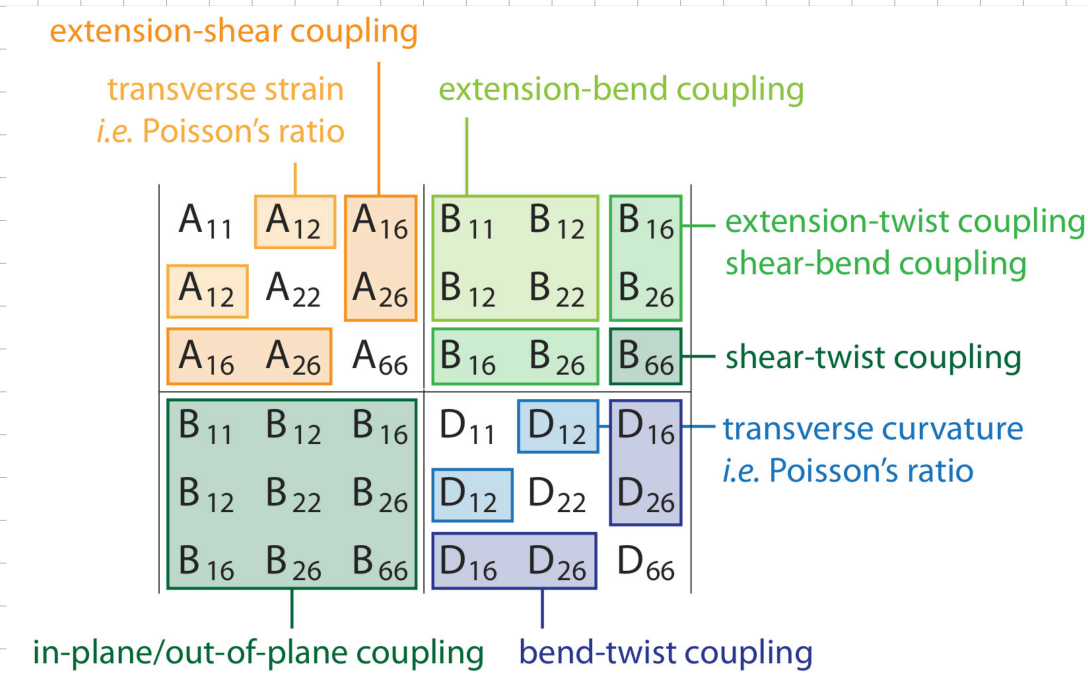
$$= \dots$$

$$= \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(\frac{t_k^3}{12} + t_k \bar{z}_k^2\right)$$

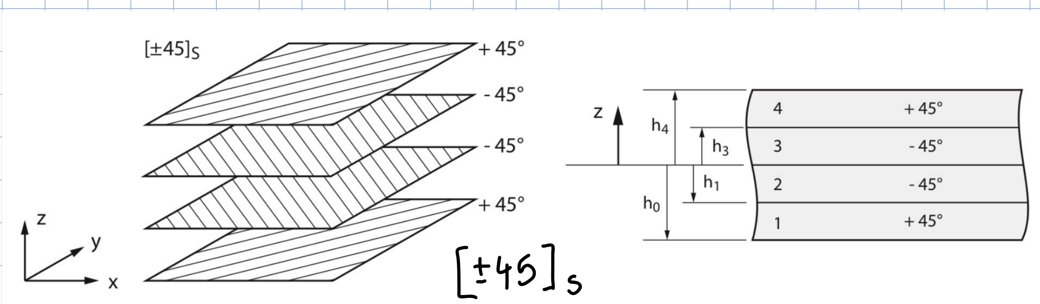
→ so for ↑ bending stiffness we want ↑ $|\bar{Q}|$ and these stiff plies far from the midplane

second moment of area of ply : $t_k^3/12$

parallel axis effect from midplane : $t_k \bar{z}_k^2$



Symmetric & Balanced Laminate : $B_{i,j} = A_{16} = A_{26} = 0$

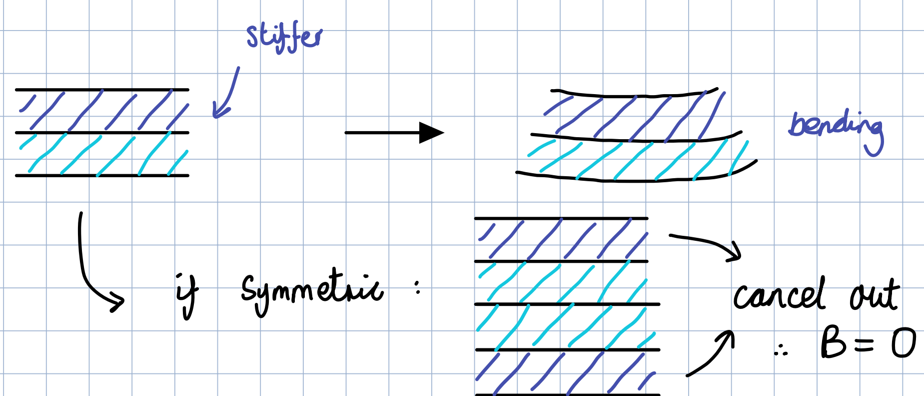


→ Symmetric :

$B_{i,j} = 0$

↳ ∴ good for manufacture as does not bend / warp with temp change

Visual Explanation :



Math Explaination :

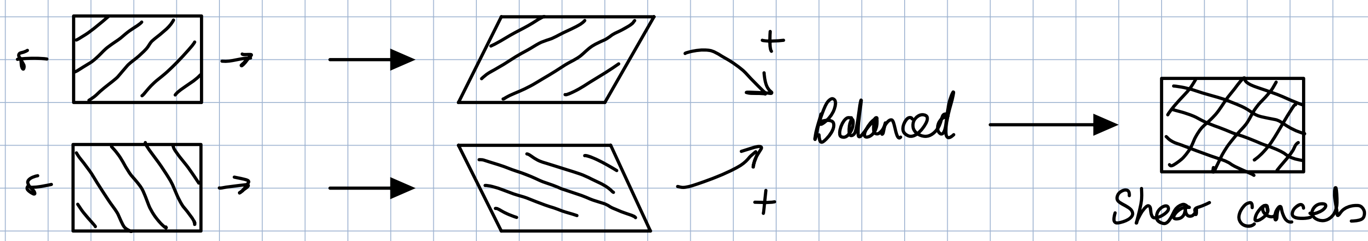
$$B_{i,j} = \frac{1}{2}\sum_{k=1}^{n}(\bar{Q}_{i,j})_k\,(h_k^2 - h_{k-1}^2)$$

this term = 0 for symmetric ∴ $B = 0$

→ Balanced : each $+\theta$ ply balanced by $-\theta$ ply of equal stiffness & thickness.

$A_{16} = A_{26} = 0$

Visual :



Math :

terms $\bar{Q}_{16}$ and $\bar{Q}_{26}$ are *odd* functions of θ

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66}) \sin\theta \cos^3\theta - (Q_{22} - Q_{12} - 2Q_{66}) \cos\theta \sin^3\theta$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66}) \cos\theta \sin^3\theta - (Q_{22} - Q_{12} - 2Q_{66}) \sin\theta \cos^3\theta$$

therefore:

$$\bar{Q}_{16}(-\theta) = -\bar{Q}_{16}(\theta) \qquad \bar{Q}_{26}(-\theta) = -\bar{Q}_{26}(\theta)$$

∴ if one $+\theta$ & one $-\theta$ $\bar{Q}$s cancel

thus, $A_{16} = A_{26} = 0$ for a balanced laminate

ABD terms vary with coordinate system (e.g changing angle) :

coordinate transformation (strain and curvature are tensors):

$$\boldsymbol{A}' = \boldsymbol{TART}^{-1}\boldsymbol{R}^{-1}$$

$$\boldsymbol{B}' = \boldsymbol{TBRT}^{-1}\boldsymbol{R}^{-1}$$

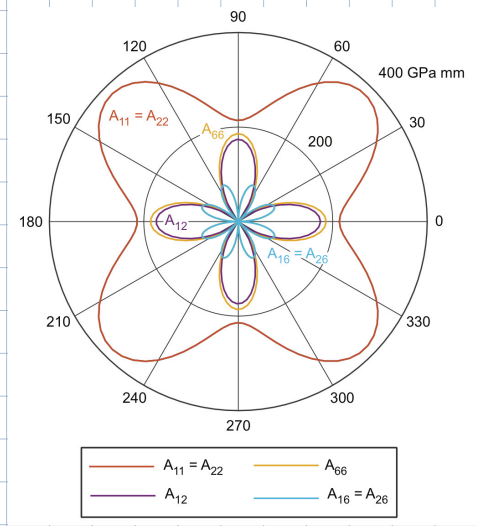
$$\boldsymbol{D}' = \boldsymbol{TDRT}^{-1}\boldsymbol{R}^{-1}$$

↳ T : transformation matrix for angle θ

→ Can plot anisotropy of laminate on polar plot :

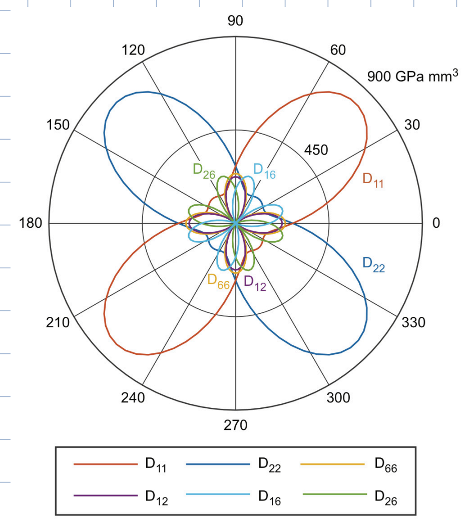
e.g for $[\pm 45]_s$

A



$B = 0$

D



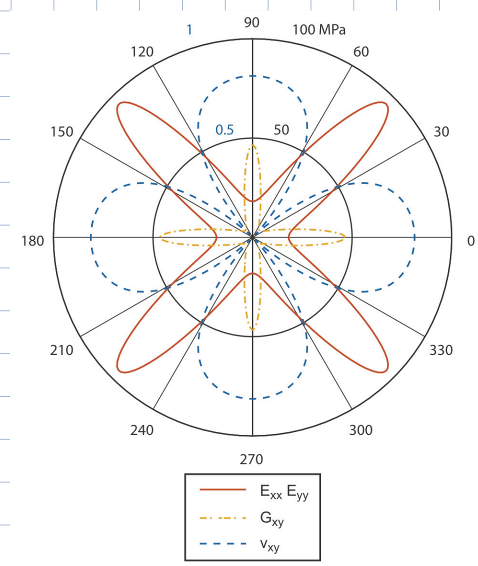
laminate **engineering constants** from compliance matrix ***a*** → $a \neq A^{-1}$

$$E_{xx} = \frac{1}{a_{11} h} \qquad E_{yy} = \frac{1}{a_{22} h}$$

$$\nu_{xy} = -\frac{a_{12}}{a_{11}} \qquad G_{xy} = \frac{1}{a_{66} h}$$

averaged through-thickness laminate stresses (σ_{xx}, σ_{yy}, τ_{xy})

divide in-plane stress resultants (N_{xx}, N_{yy}, N_{xy}) by laminate thickness h

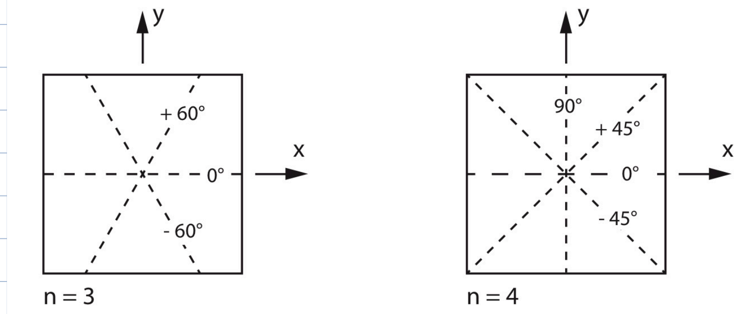


Quasi-Isotropic Laminate : isotropic in-plane properties , $B \neq 0$

$$A_{ij} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & \frac{(A_{11}-A_{12})}{2} \end{bmatrix}$$

→ Only A_{11} & A_{12} independent

→ Achieved by n plies of equal Q_{ij} and t oriented at equally-spaced angles π/n



[0/±60] [0/±45/90] etc.

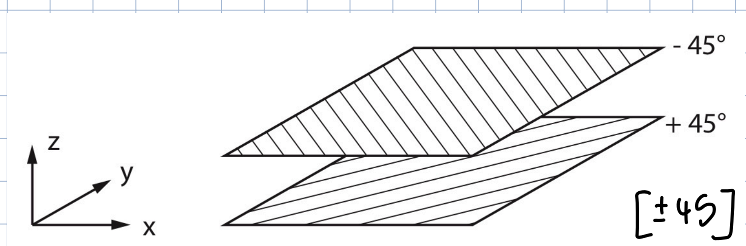
→ Avg. stiffness of anisotropic laminate = quasi-isotropic stiffness.

↳ Stiffness tailoring :

- Gain in stiffness in one direction balanced by decrease in another
- Quasi-iso laminate provides baseline stiffness

Anti-Symmetric Laminate : each +θ ply above midplane balanced by −θ below midplane

↳ ≠ asymmetric



[±45]

$A_{16} = A_{26} = 0$

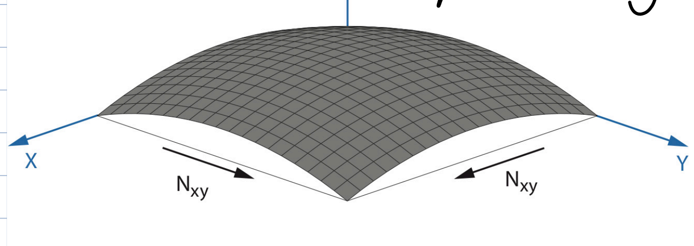
Generally, $B \neq 0$

$D_{16} = D_{26} = 0$

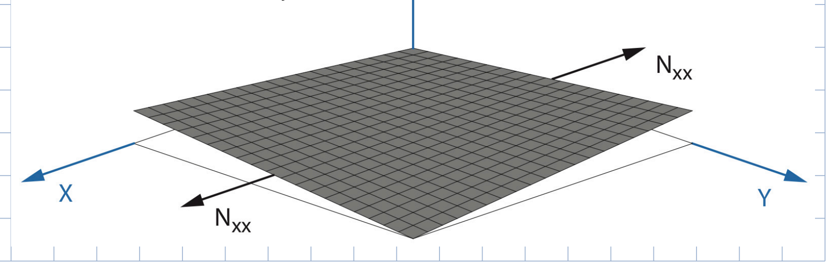
↳ no bend-twist coupling

$$
\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ \hline M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & 0 & 0 & 0 & B_{16} \\ A_{12} & A_{22} & 0 & 0 & 0 & B_{26} \\ 0 & 0 & A_{66} & B_{16} & B_{26} & 0 \\ \hline 0 & 0 & B_{16} & D_{11} & D_{12} & 0 \\ 0 & 0 & B_{26} & D_{12} & D_{22} & 0 \\ B_{16} & B_{26} & 0 & 0 & 0 & D_{66} \end{array}\right] \begin{bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \\ \hline \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}
$$

Coupling of in-plane shear & out-of-plane bending (B_{16}, B_{26})



Coupling of in-plane strain & out-of-plane twist (B_{16}, B_{26})



Two options to achieve $D_{16} = D_{26} = 0$:

→ Anti-symmetric
→ Cross-ply laminate (all layers at 0° & 90°)

compliance matrix: calculate strains and curvatures for applied loads

$$\begin{bmatrix} \varepsilon^0 \\ \kappa \end{bmatrix} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix}$$

general inversion $(ABD)^{-1}$ done by inverting sub-matrices

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MATLAB

```
>> inv(ABD)
```

↪ top left quadrant = a, etc.

Composites cured at ↑ temp and as they heat / cool they expand / contract according to their CTE.

anisotropic coefficients of thermal expansion (CTE) α_{11}, α_{22} → coefficients different along and transverse to fibres.

$$\varepsilon_{11}^T = \alpha_{11}\Delta T$$

$$\varepsilon_{22}^T = \alpha_{22}\Delta T$$

α_{11} often orders of magnitude less than α_{22}

CTEs in structural coordinate system:

$$\begin{pmatrix} \alpha_{xx} \\ \alpha_{yy} \\ \alpha_{xy} \end{pmatrix} = \boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^{-1} \begin{pmatrix} \alpha_{11} \\ \alpha_{22} \\ 0 \end{pmatrix}$$

ply strains consists of mechanical and thermal components

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \boxed{\begin{pmatrix} \varepsilon_{xx}^M \\ \varepsilon_{yy}^M \\ \gamma_{xy}^M \end{pmatrix}} + \boxed{\begin{pmatrix} \varepsilon_{xx}^T \\ \varepsilon_{yy}^T \\ \gamma_{xy}^T \end{pmatrix}}$$

$$= \boxed{\begin{pmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix}} + \boxed{\begin{pmatrix} \alpha_{xx}\Delta T \\ \alpha_{yy}\Delta T \\ \alpha_{xy}\Delta T \end{pmatrix}}$$

inverted for ply stiffness equations

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} - \alpha_{xx}\Delta T \\ \varepsilon_{yy} - \alpha_{yy}\Delta T \\ \gamma_{xy} - \alpha_{xy}\Delta T \end{pmatrix}$$

effective loads due to temp change

combined into laminate:

$$\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \varepsilon^0 \\ \kappa \end{bmatrix} - \boxed{\begin{bmatrix} \boldsymbol{N}^T \\ \boldsymbol{M}^T \end{bmatrix}}$$

thermal loads and moments:

$$\left[\boldsymbol{N}^T\right] = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (\alpha)_k \left(h_k - h_{k-1}\right) \Delta T$$

$$\left[\boldsymbol{M}^T\right] = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (\alpha)_k \left(h_k^2 - h_{k-1}^2\right) \Delta T$$

$\hookrightarrow$ if $B = 0$, $[M_T] = 0$, $[N_T] \neq 0$

Thermal loads may result in residual stresses due to different CTE's of plies.